UNIT 3: MODELING AND ANALYZING QUADRATIC FUNCTIONS

This unit investigates quadratic functions. Students study the structure of quadratic expressions and write quadratic expressions in equivalent forms. They solve quadratic equations by inspection, by completing the square, by factoring, and by using the quadratic formula. Students also graph quadratic functions and analyze characteristics of those functions, including end behavior. They write functions for various situations and build functions from other functions.

Interpret the Structure of Expressions

MGSE9-12.A.SSE.2 Use the structure of an expression to rewrite it in different equivalent forms. For example, see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 - y^2)(x^2 + y^2)$.

KEY IDEAS

1. An ***algebraic expression*** contains variables, numbers, and operation symbols.
2. A ***term*** in an algebraic expression can be a constant, a variable, or a constant multiplied by a variable or variables. Every term is separated by a plus sign.

   Example:

   The terms in the expression $5x^2 - 3x + 8$ are $5x^2$, $-3x$, and 8.

3. A ***coefficient*** is the constant number that is multiplied by a variable in a term.

   Example:

   The coefficient in the term $7x^2$ is 7.

4. A ***common factor*** is a variable or number that terms can by divided by without a remainder.

   Example:

   The common factors of $30x^2$ and $6x$ are 1, 2, 3, 6, and x.

5. A ***common factor of an expression*** is a number or term that the entire expression can be divided by without a remainder.

   Example:

   The common factor for the expression $3x^2 + 6x - 15$ is 3 because $3x^2 + 6x - 15 = 3(x^2 + 2x - 5)$.

6. If parts of an expression are independent of each other, the expression can be interpreted in different ways.

   Example:

   In the expression $h(3x^2 + 6x - 15)$, the factors h and $(3x^2 + 6x - 15)$ are independent of each other. It can be interpreted as the product of h and a term that does not depend on h.

7. The structure of some expressions can be used to help rewrite them. For example, some fourth-degree expressions are in quadratic form.

   Example:
   $x^2 + 5x + 4 = (x + 4)(x + 1)$

   Example:
   $x^2 - y^2 = (x - y)(x + y)$

REVIEW EXAMPLES

1. Consider the expression $3n^2 + n + 2$.

   a. What is the coefficient of n?

   b. What terms are being added in the expression?

   Solution:

   a. 1

   b. $3n^2$, n, and 2

2. Factor the expression $16a^2 - 81$.

   Solution:
   The expression $16a^2 - 81$ is quadratic in form because it is the difference of two squares ($16a^2 = (4a)^2$ and $81 = 9^2$) and both terms of the binomial are perfect squares. The difference of squares can be factored as:

   $x^2 - y^2 = (x + y)(x - y)$

$16a^2 - 81$	Original expression
$(4a + 9)(4a - 9)$	Factor the binomial (difference of two squares).

3. Factor the expression $12x^2 + 14x - 6$.

   Solution:

$12x^2 + 14x - 6$	Original expression
$2(6x^2 + 7x - 3)$	Factor the trinomial (common factor).
$2(3x - 1)(2x + 3)$	Factor.

SAMPLE ITEMS

1. **Which expression is equivalent to $121x^2 - 64y^2$?**
   - **A.** $(11x - 16y)(11x + 16y)$
   - **B.** $(11x - 16y)(11x - 16y)$
   - **C.** $(11x + 8y)(11x + 8y)$
   - **D.** $(11x + 8y)(11x - 8y)$

Correct Answer: D

2. **What is a common factor for the expression $24x^2 + 16x + 144$?**
   - **A.** 16
   - **B.** $8x$
   - **C.** $3x^2 + 2x + 18$
   - **D.** $8(x - 2)(3x^2 + 9)$

Correct Answer: C

3. **Which of these shows the complete factorization of $6x^2y^2 - 9xy - 42$?**
   - **A.** $3(2xy^2 - 7)(xy^2 + 2)$
   - **B.** $(3xy + 6)(2xy - 7)$
   - **C.** $3(2xy - 7)(xy + 2)$
   - **D.** $(3xy^2 + 6)(2xy^2 - 7)$

Correct Answer: C

Write Expressions in Equivalent Forms to Solve Problems

MGSE9-12.A.SSE.3 Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.

MGSE9-12.A.SSE.3a Factor any quadratic expression to reveal the zeros of the function defined by the expression.

MGSE9-12.A.SSE.3b Complete the square in a quadratic expression to reveal the maximum and minimum value of the function defined by the expression.

KEY IDEAS

1. The ***zeros***, ***roots***, or ***x-intercepts*** of a function are the values of the variable that make the function equal to zero. When the function is written in factored form, the Zero Product Property can be used to find the zeros of the function. The Zero Product Property states that if the product of two factors is zero, then one or both of the factors must be zero. So, the zeros of the function are the values that make either factor equal to zero.

   Example:

$x^2 - 7x + 12 = 0$	Original equation
$(x - 3)(x - 4) = 0$	Factor.

   Set each factor equal to zero and solve.

$x - 3 = 0$	$x - 4 = 0$
$x = 3$	$x = 4$

   The zeros of the function $y = x^2 - 7x + 12$ are $x = 3$ and $x = 4$.

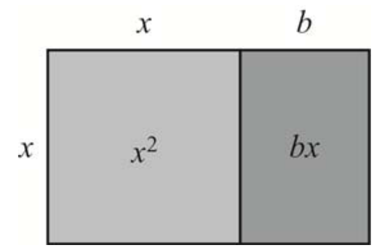
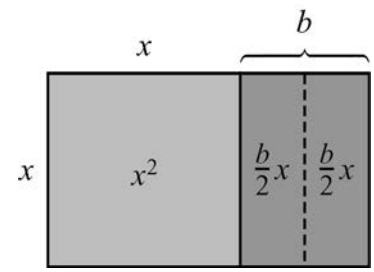
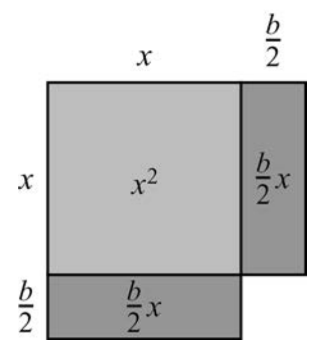
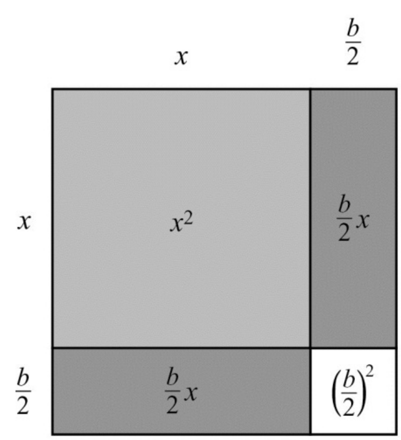
2. To ***complete the square*** of a quadratic function means to write a function as the square of a sum. The standard form for a quadratic expression is $\boldsymbol{ax^2 + bx + c}$, where $a \neq 0$. When $a = 1$, completing the square of the function $x^2 + bx = d$ gives $\left(x + \frac{b}{2}\right)^2 = d + \left(\frac{b}{2}\right)^2$. To complete the square when the value $a \neq 1$, factor the value of a from the expression.

   Example:

   To complete the square, take half of the coefficient of the x-term, square it, and add it to both sides of the equation.

$x^2 + bx = d$	Original expression
$x^2 + bx + \left(\frac{b}{2}\right)^2 = d + \left(\frac{b}{2}\right)^2$	The coefficient of x is b. Half of b is $\frac{b}{2}$. Add the square of $\frac{b}{2}$ to both sides of the equation.
$\left(x + \frac{b}{2}\right)^2 = d + \left(\frac{b}{2}\right)^2$	The expression on the left side of the equation is a perfect square trinomial. Factor to write it as a binomial squared.

This figure shows how a model can represent completing the square of the expression $x^2 + bx$, where b is positive.

This model represents the expression $x^2 + bx$. To complete the square, create a model that is a square.	
Split the rectangle for bx into two rectangles that represent $\frac{b}{2}x$.	
Rearrange the two rectangles that represent $\frac{b}{2}x$.	
The missing piece of the square measures $\frac{b}{2}$ by $\frac{b}{2}$. Add and subtract $\left(\frac{b}{2}\right)^2$ to complete the model of the square. The large square has a side length of $x + \frac{b}{2}$, so this model represents $\left(x + \frac{b}{2}\right)^2$. $x^2 + bx = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$.	

Examples:

Complete the square:

$x^2 + 3x + 7$

$\left(x^2 + 3x + \left(\frac{3}{2}\right)^2\right) + 7 - \left(\frac{3}{2}\right)^2$

$\left(x + \frac{3}{2}\right)^2 + \frac{19}{4}$

Complete the square:

$x^2 + 3x + 7 = 0$

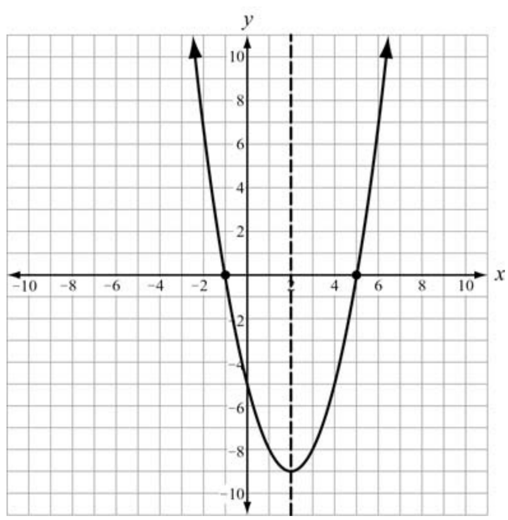
$x^2 + 3x + \left(\frac{3}{2}\right)^2 = -7 + \left(\frac{3}{2}\right)^2$

$\left(x + \frac{3}{2}\right)^2 = -\frac{19}{4}$

3. Every quadratic function has a ***minimum*** or a ***maximum***. This minimum or maximum is located at the ***vertex*** (h, k). The vertex (h, k) also identifies the ***axis of symmetry*** and the minimum or maximum value of the function. The axis of symmetry is $x = h$.

Example:

The quadratic equation $f(x) = x^2 - 4x - 5$ is shown in this graph. The minimum of the function occurs at the vertex $(2, -9)$. The zeros or x-intercepts of the function are $(-1, 0)$ and $(5, 0)$. The axis of symmetry is $x = 2$.



4. The ***vertex form*** of a quadratic function is $f(x) = a(x - h)^2 + k$ where (h, k) is the vertex. One way to convert an equation from standard form to vertex form is to complete the square.
5. The vertex of a quadratic function can also be found by using the ***standard form*** and determining the value $\frac{-b}{2a}$. The vertex is $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$.

Important Tips

- When you complete the square, make sure you are only changing the form of the expression and not changing the value.
- When completing the square in an **expression**, add **and subtract** half of the coefficient of the x-term squared.
- When completing the square in an **equation**, add half of the coefficient of the x-term squared **to both sides of the equation.**

REVIEW EXAMPLES

1. Write $f(x) = 2x^2 + 12x + 1$ in vertex form.

Solution Method 1:
The function is in standard form, where $a = 2$, $b = 12$, and $c = 1$.

$2x^2 + 12x + 1$	Original expression
$2(x^2 + 6x) + 1$	Factor out 2 from the quadratic and linear terms.
$2\left(x^2 + 6x + \left(3\right)^2 - \left(3\right)^2\right) + 1$	Add and subtract the square of half of the coefficient of the linear term.
$2\left(x^2 + 6x + \left(3\right)^2\right) - 2(9) + 1$	Remove the subtracted term from the parentheses. Remember to multiply by a.
$2\left(x^2 + 6x + \left(3\right)^2\right) - 17$	Combine the constant terms.
$2\left(x + 3\right)^2 - 17$	Write the perfect square trinomial as a binomial squared.

The vertex of the function is (–3, –17).

Solution Method 2:
The vertex of a quadratic function can also be found by writing the polynomial in standard form and determining the value of $\frac{-b}{2a}$. The vertex is $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$.

For $f(x) = 2x^2 + 12x + 1$, $a = 2$, $b = 12$, and $c = 1$.

$$\frac{-b}{2a} = \frac{-12}{2(2)} = \frac{-12}{4} = -3$$

$$\begin{aligned} f(-3) &= 2(-3)^2 + 12(-3) + 1 \\ &= 2(-3)^2 - 36 + 1 \\ &= 18 - 36 + 1 \end{aligned}$$

The vertex of the function is (–3, –17).

2. The function $h(t) = -t^2 + 8t + 2$ represents the height, in feet, of a stream of water being squirted out of a fountain after t seconds. What is the maximum height of the water?

Solution:

The function is in standard form, where $a = -1$, $b = 8$, and $c = 2$.

The x-coordinate of the vertex is $\frac{-b}{2a} = \frac{-8}{2(-1)} = 4$.

The y-coordinate of the vertex is $h(4) = -(4)^2 + 8(4) + 2 = 18$.

The vertex of the function is (4, 18). So, the maximum height of the water occurs at 4 seconds and is 18 feet.

3. What are the zeros of the function represented by the quadratic expression $x^2 + 6x - 27$?

Solution:

Factor the expression: $x^2 + 6x - 27 = (x + 9)(x - 3)$.

	x + 9	
x – 3	x^2	$9x$
	$-3x$	-27

Set each factor equal to 0 and solve for x.

$x + 9 = 0$ $\qquad$ $x - 3 = 0$

$x = -9$ $\qquad$ $x = 3$

The zeros are $x = -9$ and $x = 3$. This means that $f(-9) = 0$ and $f(3) = 0$.

4. What are the zeros of the function represented by the quadratic expression $2x^2 - 5x - 3$?

Solution:

Factor the expression: $2x^2 - 5x - 3 = (2x + 1)(x - 3)$.

Set each factor equal to 0 and solve for x.

$2x + 1 = 0$ $\qquad$ $x - 3 = 0$

$x = -\frac{1}{2}$ $\qquad$ $x = 3$

The zeros are $x = -\frac{1}{2}$ and $x = 3$.

SAMPLE ITEMS

1. What are the zeros of the function represented by the quadratic expression $2x^2 + x - 3$?

A. $x = -\frac{3}{2}$ and $x = 1$

B. $x = -\frac{2}{3}$ and $x = 1$

C. $x = -1$ and $x = \frac{2}{3}$

D. $x = -1$ and $x = -\frac{3}{2}$

Correct Answer: A

2. What is the vertex of the graph of $f(x) = x^2 + 10x - 9$?

A. (5, 66)
B. (5, –9)
C. (–5, –9)
D. (–5, –34)

Correct Answer: D

3. Which of these is the result of completing the square for the expression $x^2 + 8x - 30$?

A. $(x + 4)^2 - 30$
B. $(x + 4)^2 - 46$
C. $(x + 8)^2 - 30$
D. $(x + 8)^2 - 94$

Correct Answer: B

4. The expression $-x^2 + 70x - 600$ represents a company's profit for selling x items. For which number(s) of items sold is the company's profit equal to \$0?

A. 0 items
B. 35 items
C. 10 items and 60 items
D. 20 items and 30 items

Correct Answer: C

Create Equations That Describe Numbers or Relationships

MGSE9-12.A.CED.1 Create equations and inequalities in one variable and use them to solve problems. Include equations arising from linear, quadratic, and exponential functions (integer inputs only).

MGSE9-12.A.CED.2 Create linear, quadratic, and exponential equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. (The phrase "in two or more variables" refers to formulas like the compound interest formula, in which $A = P\left(1 + \frac{r}{n}\right)^{nt}$ has multiple variables.)

MGSE9-12.A.CED.4 Rearrange formulas to highlight a quantity of interest using the same reasoning as in solving equations. *Examples: Rearrange Ohm's law $V = IR$ to highlight resistance R; Rearrange area of a circle formula $A = \pi r^2$ to highlight the radius r.*

KEY IDEAS

1. Quadratic equations can be written to model real-world situations.

   Here are some examples of real-world situations that can be modeled by quadratic functions:

   - Finding the area of a shape: Given that the length of a rectangle is 5 units more than the width, the area of the rectangle in square units can be represented by $A = x(x + 5)$, where x is the width and $x + 5$ is the length.
   - Finding the product of consecutive integers: Given a number, n, the next consecutive number is $n + 1$ and the next consecutive even (or odd) number is $n + 2$. The product, P, of two consecutive numbers is $P = n(n + 1)$.
   - Finding the height of a projectile that is dropped: When heights are given in metric units, the equation used is $h(t) = -4.9t^2 + v_o t + h_o$, where v_o is the initial velocity, in meters per second, and h_o is the initial height, in meters. The coefficient –4.9 represents half the force of gravity. When heights are given in customary units, the equation used is $h(t) = -16t^2 + v_o t + h_o$, where v_o is the initial velocity, in feet per second, and h_o is the initial height, in feet. The coefficient –16 represents half the force of gravity. For example, the height, in feet, of a ball thrown with an initial velocity of 60 feet per second and an initial height of 4 feet can be represented by $h(t) = -16t^2 + 60t + 4$, where t is seconds.

2. You can use the properties of equality to solve for a variable in an equation. Use inverse operations on both sides of the equation until you have isolated the variable.

   Example:

   What is the value of r when $S = 0$ for the equation $S = 2\pi r^2 + 2\pi rh$ for r?

Solution:

First, factor the expression $2\pi r^2 + 2\pi rh$.

$$2\pi r(r + h)$$

Next, set each factor equal to 0.

$$\begin{aligned} 2\pi r &= 0, \qquad & r + h &= 0 \\ r &= 0, \qquad & r &= -h \end{aligned}$$

3. To graph a quadratic equation, find the vertex of the graph and the zeros of the equation. The axis of symmetry goes through the vertex and divides the graph into two sides that are mirror images of each other. To draw the graph, you can plot points on one side of the parabola and use symmetry to find the corresponding points on the other side of the parabola.

Example:

Graph the quadratic equation $y = x^2 + 5x + 6$.

Solution:

First, we can find the zeros by solving for x when $y = 0$. This is where the graph crosses the x-axis.

$$\begin{aligned} 0 &= x^2 + 5x + 6 \\ 0 &= (x + 2)(x + 3) \\ x + 2 &= 0,\ x + 3 = 0 \\ x &= -2, \quad x = -3\text{; this gives us the points } (-2, 0) \text{ and } (-3, 0). \end{aligned}$$

Next, we can find the axis of symmetry by finding the vertex. The axis of symmetry is the equation $x = -\frac{b}{2a}$. To find the vertex, we first find the axis of symmetry.

$$x = -\frac{5}{2(1)} = -\frac{5}{2}$$

Now we can find the value of the y-coordinate of the vertex.

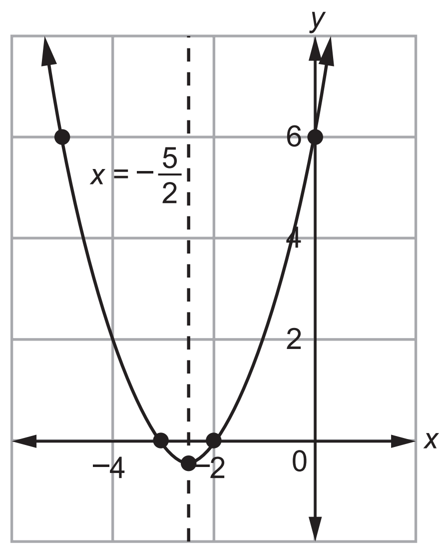
$$\begin{aligned} y &= \left(-\frac{5}{2}\right)^2 + 5\left(-\frac{5}{2}\right) + 6 \\ &= \frac{25}{4} + \left(-\frac{25}{2}\right) + 6 \\ &= \frac{25}{4} + -\frac{50}{4} + \frac{24}{4} \\ &= -\frac{25}{4} + \frac{24}{4} \\ &= -\frac{1}{4} \end{aligned}$$

So, the vertex is located at $\left(-\frac{5}{2}, -\frac{1}{4}\right)$.

Next, we can find two more points to continue the curve. We can use the *y*-intercept to find the first of the two points.

$y = (0)^2 + 5(0) + 6 = 6$. The *y*-intercept is at (0, 6).

This point is 2.5 more than the axis of symmetry, so the last point will be 2.5 less than the axis of symmetry. The point 2.5 less than the axis of symmetry with a *y*-value of 6 is (–5, 6).



4. The axis of symmetry is the midpoint for each corresponding pair of *x*-coordinates with the same *y*-value. If (x_1, y) is a point on the graph of a parabola and $x = h$ is the axis of symmetry, then (x_2, y) is also a point on the graph, and x_2 can be found using this equation: $\frac{x_1 + x_2}{2} = h$. In the example above, we can use the zeros (–3, 0) and (–2, 0) to find the axis of symmetry.

$$\frac{-3 + -2}{2} = -\frac{5}{2} = -2.5\text{, so } x = -2.5$$

REVIEW EXAMPLES

1. The product of two consecutive positive integers is 132.

   a. Write an equation to model the situation.

   b. What are the two consecutive integers?

   Solution:

   a. Let *n* represent the lesser of the two integers. Then *n* + 1 represents the greater of the two integers. So, the equation is $n(n + 1) = 132$.

   b. Solve the equation for *n*.

   $n(n + 1) = 132$ Original equation

$n^2 + n = 132$ Distributive Property

$n^2 + n - 132 = 0$ Subtraction Property of Equality

$(n + 12)(n - 11) = 0$ Factor.

Set each factor equal to 0 and solve for n.

$n + 12 = 0$ $n - 11 = 0$

$n = -12$ $n = 11$

Because the two consecutive integers are both positive, $n = -12$ cannot be the solution. So, $n = 11$ is the solution, which means that the two consecutive integers are 11 and 12.

2. The formula for the volume of a cylinder is $V = \pi r^2 h$.

   a. Solve the formula for r.

   b. If the volume of a cylinder is 200π cubic inches and the height of the cylinder is 8 inches, what is the radius of the cylinder?

   Solution:

   a. Solve the formula for r.

   $V = \pi r^2 h$ Original formula

   $\frac{V}{\pi h} = r^2$ Division Property of Equality

   $\pm\sqrt{\frac{V}{\pi h}} = r$ Take the square root of both sides.

   $\sqrt{\frac{V}{\pi h}} = r$ Choose the positive value because the radius cannot be negative.

   b. Substitute 200π for V and 8 for h and evaluate.

   $$r = \sqrt{\frac{V}{\pi h}} = \sqrt{\frac{200\pi}{\pi(8)}} = \sqrt{\frac{200}{8}} = \sqrt{25} = 5$$

   The radius of the cylinder is 5 inches.

3. Graph the function represented by the equation $y = 3x^2 - 6x - 9$.

Solution:

Find the zeros of the equation.

$0 = 3x^2 - 6x - 9$ Set the equation equal to 0.

$0 = 3(x^2 - 2x - 3)$ Factor out 3.

$0 = 3(x - 3)(x + 1)$ Factor.

$0 = (x - 3)(x + 1)$ Division Property of Equality

Set each factor equal to 0 and solve for x.

$x - 3 = 0$ $\quad$ $x + 1 = 0$

$x = 3$ $\quad$ $x = -1$

The zeros are at $x = -1$ and $x = 3$.

Find the vertex of the graph.

$$\frac{-b}{2a} = \frac{-(-6)}{2(3)} = \frac{6}{6} = 1$$

Substitute 1 for x in the original equation to find the y-value of the vertex:

$$\begin{aligned} 3(1)^2 - 6(1) - 9 &= 3 - 6 - 9 \\ &= -12 \end{aligned}$$

Graph the two x-intercepts (3, 0) and (–1, 0) and the vertex (1, –12).

Another descriptive point is the y-intercept. You can find the y-intercept by substituting 0 for x.

$$y = 3x^2 - 6x - 9$$

$$y = 3(0)^2 - 6(0) - 9$$

$$y = -9$$

You can find more points for your graph by substituting x-values into the function. Find the y-value when $x = -2$.

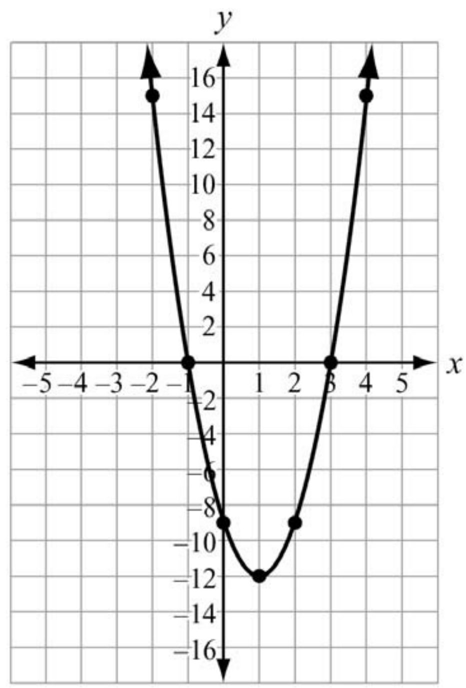
$$y = 3x^2 - 6x - 9$$

$$y = 3(-2)^2 - 6(-2) - 9$$

$$y = 3(4) + 12 - 9$$

$$y = 15$$

Graph the points (0, –9) and (–2, 15). Then use the concept of symmetry to draw the rest of the function. The axis of symmetry is $x = 1$. So, the mirror image of (0, –9) is (2, –9) and the mirror image of (–2, 15) is (4, 15).



SAMPLE ITEMS

1. **A garden measuring 8 feet by 12 feet will have a walkway around it. The walkway has a uniform width, and the area covered by the garden and the walkway is 192 square feet. What is the width of the walkway?**

   A. 2 feet
   B. 3.5 feet
   C. 4 feet
   D. 6 feet

Correct Answer: A

2. **The formula for the area of a circle is $A = \pi r^2$. Which equation shows the formula in terms of r?**

   A. $r = \frac{2A}{\pi}$

   B. $r = \frac{\sqrt{A}}{\pi}$

   C. $r = \sqrt{\frac{A}{\pi}}$

   D. $r = \frac{A}{2\pi}$

Correct Answer: C

Solve Equations and Inequalities in One Variable

MGSE9-12.A.REI.4 Solve quadratic equations in one variable.

MGSE9-12.A.REI.4a Use the method of completing the square to transform any quadratic equation in x into an equation of the form $(x - p)^2 = q$ that has the same solutions. Derive the quadratic formula from $ax^2 + bx + c = 0$.

MGSE9-12.A.REI.4b Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, factoring, completing the square, and the quadratic formula, as appropriate to the initial form of the equation (limit to real number solutions).

KEY IDEAS

1. When quadratic equations do not have a linear term, you can solve the equation by taking the ***square root*** of each side of the equation. Remember, every square root has a positive value and a negative value. Earlier in the guide, we eliminated the negative answers when they represented length or distance.

   Example:

   $3x^2 - 147 = 0$

$3x^2 = 147$	Addition Property of Equality
$x^2 = 49$	Multiplicative Inverse Property
$x = \pm 7$	Take the square root of both sides.

   Check your answers:

   $$\begin{aligned} 3(7)^2 - 147 &= 3(49) - 147 \\ &= 147 - 147 \\ &= 0 \end{aligned}$$

   $$\begin{aligned} 3(-7)^2 - 147 &= 3(49) - 147 \\ &= 147 - 147 \\ &= 0 \end{aligned}$$

2. You can ***factor*** some quadratic equations to find the solutions. To do this, rewrite the equation in standard form set equal to zero ($ax^2 + bx + c = 0$). Factor the expression, set each factor to 0 (by the Zero Product Property), and then solve for x in each resulting equation. This will provide two rational values for x.

   Example:

   $x^2 - x = 12$

$x^2 - x - 12 = 0$	Addition Property of Equality
$(x - 4)(x + 3) = 0$	Factor.

   Set each factor equal to 0 and solve.

$x - 4 = 0$	$x + 3 = 0$
$x = 4$	$x = -3$

Check your answers:

$$4^2 - 4 = 16 - 4 \qquad (-3)^2 - (-3) = 9 + 3$$
$$= 12 \qquad\qquad = 12$$

3. You can complete the square to solve a quadratic equation. First, write the expression that represents the function in standard form, $ax^2 + bx + c = 0$. Subtract the constant from both sides of the equation: $ax^2 + bx = -c$. Divide both sides of the equation by a: $x^2 + \frac{b}{a}x = \frac{-c}{a}$. Add the square of half the coefficient of the x-term to both sides: $x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \frac{-c}{a} + \left(\frac{b}{2a}\right)^2$. Write the perfect square trinomial as a binomial squared: $\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$. Take the square root of both sides of the equation and solve for x.

Example:

$5x^2 - 6x - 8 = 0$	
$5x^2 - 6x = 8$	Addition Property of Equality
$x^2 - \frac{6}{5}x = \frac{8}{5}$	Division Property of Equality
$x^2 - \frac{6}{5}x + \left(\frac{-3}{5}\right)^2 = \frac{8}{5} + \left(\frac{3}{5}\right)^2$	Addition Property of Equality
$x^2 - \frac{6}{5}x + \left(\frac{-3}{5}\right)^2 = \frac{8}{5} + \frac{9}{25}$	
$\left(x - \frac{3}{5}\right)^2 = \frac{40}{25} + \frac{9}{25}$	Distribution Property
$\left(x - \frac{3}{5}\right)^2 = \frac{49}{25}$	
$x - \frac{3}{5} = \pm\frac{7}{5}$	Take the square root of both sides.
$x = \frac{3}{5} \pm \frac{7}{5}$	Addition Property of Equality
$x = \frac{3}{5} + \frac{7}{5} = \frac{10}{5} = 2;\ x = \frac{3}{5} - \frac{7}{5} = -\frac{4}{5}$	Solve for x for both operations.

4. All quadratic equations can be solved using the quadratic formula. The ***quadratic formula*** is $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, where $ax^2 + bx + c = 0$. The quadratic formula will yield real solutions. We can solve the previous equation using the quadratic formula.

Example:

$5x^2 - 6x - 8 = 0$, where $a = 5$, $b = -6$, and $c = -8$.

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(5)(-8)}}{2(5)}$$

$$x = \frac{6 \pm \sqrt{36 - 4(-40)}}{10}$$

$$x = \frac{6 \pm \sqrt{36 - (-160)}}{10}$$

$$x = \frac{6 \pm \sqrt{36 + (160)}}{10}$$

$$x = \frac{6 \pm \sqrt{196}}{10}$$

$$x = \frac{6 \pm 14}{10}$$

$$x = \frac{6 + 14}{10} = \frac{20}{10} = 2; \quad x = \frac{6 - 14}{10} = \frac{-8}{10} = \frac{-4}{5}$$

Important Tip

✍ While there may be several methods that can be used to solve a quadratic equation, some methods may be easier than others for certain equations.

REVIEW EXAMPLES

1. Solve the equation $x^2 - 10x + 25 = 0$ by factoring.

   Solution:
   Factor: $x^2 - 10x + 25 = (x - 5)(x - 5)$.

   Both factors are the same, so solve the equation:

   $x - 5 = 0$

   $x = 5$

2. Solve the equation $x^2 - 100 = 0$ by using square roots.

   Solution:
   Solve the equation using square roots.

   $x^2 = 100$ Add 100 to both sides of the equation.

   $x = \pm\sqrt{100}$ Take the square root of both sides of the equation.

   $x = \pm 10$ Evaluate.

3. Solve the equation $4x^2 - 7x + 3 = 0$ using the quadratic formula.

   Solution:
   Solve the equation using the quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, where $ax^2 - bx + c = 0$.

   Given the equation in standard form, the following values will be used in the formula:

   $a = 4$, $b = -7$, and $c = 3$

   $x = \dfrac{-(-7) \pm \sqrt{(-7)^2 - 4(4)(3)}}{2(4)}$ Substitute each value into the quadratic formula.

   $x = \dfrac{7 \pm \sqrt{1}}{8}$ Simplify the expression.

   $x = \dfrac{7+1}{8} = 1$ and $x = \dfrac{7-1}{8} = \dfrac{6}{8} = \dfrac{3}{4}$ Evaluate.

SAMPLE ITEMS

1. What are the solutions to the equation $2x^2 - 2x - 12 = 0$?

A. $x = -4, x = 3$
B. $x = -3, x = 4$
C. $x = -2, x = 3$
D. $x = -6, x = 2$

Correct Answer: C

2. What are the solutions to the equation $6x^2 - x - 40 = 0$?

A. $x = -\frac{8}{3}, x = -\frac{5}{2}$

B. $x = -\frac{8}{3}, x = \frac{5}{2}$

C. $x = \frac{5}{2}, x = \frac{8}{3}$

D. $x = -\frac{5}{2}, x = \frac{8}{3}$

Correct Answer: D

3. What are the solutions to the equation $x^2 - 5x = 14$?

A. $x = -7$, $x = -2$
B. $x = -14$, $x = -1$
C. $x = -2$, $x = 7$
D. $x = -1$, $x = 14$

Correct Answer: C

4. An object is thrown in the air with an initial velocity of 5 m/s from a height of 9 m. The equation $h(t) = -4.9t^2 + 5t + 9$ models the height of the object in meters after t seconds.

About how many seconds does it take for the object to hit the ground? Round your answer to the nearest tenth of a second.

A. 0.940 second
B. 1.50 seconds
C. 2.00 seconds
D. 9.00 seconds

Correct Answer: C

Build a Function That Models a Relationship Between Two Quantities

MGSE9-12.F.BF.1 Write a function that describes a relationship between two quantities.

> **MGSE9-12.F.BF.1a** Determine an explicit expression and the recursive process (steps for calculation) from context. *For example, if Jimmy starts out with $15 and earns $2 a day, the explicit expression "2x + 15" can be described recursively (either in writing or verbally) as "to find out how much money Jimmy will have tomorrow, you add $2 to his total today."* $J_n = J_{n-1} + 2$, $J_0 = 15$

KEY IDEAS

1. An ***explicit expression*** contains variables, numbers, and operation symbols and does not use an equal sign to relate the expression to another quantity.
2. A ***recursive process*** can show that a quadratic function has second differences that are equal to one another.

Example:

Consider the function $f(x) = x^2 + 4x - 1$.

This table of values shows five values of the function.

x	$f(x)$
–2	–5
–1	–4
0	–1
1	4
2	11

The first and second differences are shown. The first differences are the differences between the consequence terms. The second differences are the differences between the consequence terms of the first differences.

x	$f(x)$
–2	–5
–1	–4
0	–1
1	4
2	11

First differences

–4 – (–5) = 1

–1 – (–4) = 3

4 – (–1) = 5

11 – 4 = 7

Second differences

3 – 1 = 2

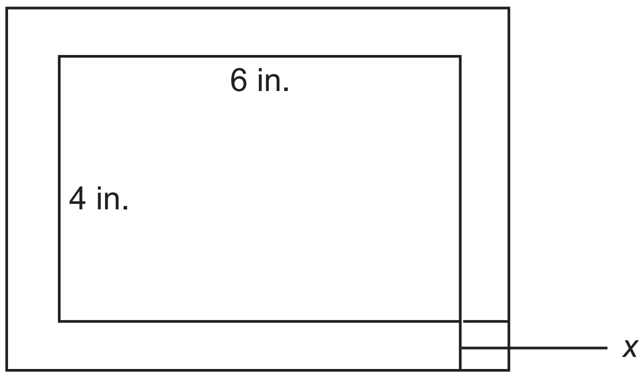
5 – 3 = 2

7 – 5 = 2

3. A ***recursive function*** is one in which each function value is based on a previous value (or values) of the function.

REVIEW EXAMPLES

1. Annie is framing a photo with a length of 6 inches and a width of 4 inches. The distance from the edge of the photo to the edge of the frame is x inches. The combined area of the photo and frame is 63 square inches.



Note: Image is NOT drawn to scale.

   a. Write a quadratic function to find the distance from the edge of the photo to the edge of the frame.
   b. How wide are the photo and frame together?

   Solution:

   a. The length of the photo and frame is $x + 6 + x = 6 + 2x$. The width of the photo and frame is $x + 4 + x = 4 + 2x$. The area of the frame is $(6 + 2x)(4 + 2x) = 4x^2 + 20x + 24$. Set this expression equal to the area: $63 = 4x^2 + 20x + 24$.
   b. Solve the equation for x.

$$63 = 4x^2 + 20x + 24$$

$$0 = 4x^2 + 20x - 39$$

$$x = -6.5 \text{ or } x = 1.5$$

   Length cannot be negative, so the distance from the edge of the photo to the edge of the frame is 1.5 inches. Therefore, the width of the photo and frame together is $4 + 2x = 4 + 2(1.5) = 7$ inches.

2. A scuba diving company currently charges \$100 per dive. On average, there are 30 customers per day. The company performed a study and learned that for every \$20 price increase, the average number of customers per day would be reduced by 2.

   a. The total revenue from the dives is the price per dive multiplied by the number of customers. What is the revenue after 4 price increases?
   b. Write a quadratic equation to represent x price increases.
   c. What price would give the greatest revenue?

Solution:

a. Make a table to show the revenue after 4 price increases.

Number of Price Increases	Price per Dive ($)	Number of Customers per Day	Revenue per Day ($)
0	100	30	(100)(30) = 3,000
1	100 + 20(1) = 120	30 – 2(1) = 28	(120)(28) = 3,360
2	100 + 20(2) = 140	30 – 2(2) = 26	(140)(26) = 3,640
3	100 + 20(3) = 160	30 – 2(3) = 24	(160)(24) = 3,840
4	100 + 20(4) = 180	30 – 2(4) = 22	(180)(22) = 3,960

The revenue after 4 price increases is ($180)(22) = $3,960.

b. The table shows a pattern. The price per dive for x price increases is $100 + 20x$. The number of customers for x price increases is $30 - 2x$. So, the equation $y = (100 + 20x)(30 - 2x) = -40x^2 + 400x + 3{,}000$ represents the revenue for x price increases.

c. To find the price that gives the greatest revenue, first find the number of price increases that gives the greatest value. This occurs at the vertex.

Use $\frac{-b}{2a}$ with $a = -40$ and $b = 400$.

$$\frac{-b}{2a} = \frac{-400}{2(-40)} = \frac{-400}{-80} = 5$$

The maximum revenue occurs after 5 price increases.

$100 + 20(5) = 200$

The price of $200 per dive gives the greatest revenue.

3. Consider the sequence 2, 6, 12, 20, 30, ...

a. What explicit expression can be used to find the next term in the sequence?

b. What is the tenth term of the sequence?

Solution:

a. The difference between terms is not constant, so the operation involves multiplication. Make a table to try to determine the relationship between the number of the term and the value of the term.

Term number	Term value	Relationship
1	2	$1 \cdot 2$
2	6	$2 \cdot 3$
3	12	$3 \cdot 4$
4	20	$4 \cdot 5$
5	30	$5 \cdot 6$

Notice the pattern: The value of each term is the product of the term number and one more than the term number. So, the expression is $n(n + 1)$ or $n^2 + n$.

b. The tenth term is $n^2 + n = (10)^2 + (10) = 110$.

SAMPLE ITEMS

1. What explicit expression can be used to find the next term in this sequence?

2, 8, 18, 32, 50, . . .

A. $2n$
B. $2n + 6$
C. $2n^2$
D. $2n^2 + 1$

Correct Answer: C

2. The function $s(t) = vt + h - 0.5at^2$ represents the height of an object, s, from the ground after the time, t, when the object is thrown with an initial velocity of v at an initial height of h and where a is the acceleration due to gravity (32 feet per second squared).

A baseball player hits a baseball 4 feet above the ground with an initial velocity of 80 feet per second. About how long will it take the baseball to hit the ground?

A. 2 seconds
B. 3 seconds
C. 4 seconds
D. 5 seconds

Correct Answer: D

3. A café's annual income depends on x, the number of customers. The function $I(x) = 4x^2 - 20x$ describes the café's total annual income. The function $C(x) = 2x^2 + 5$ describes the total amount the café spends in a year. The café's annual profit, $P(x)$, is the difference between the annual income and the amount spent in a year.

Which function describes $P(x)$?

A. $P(x) = 2x^2 - 20x - 5$
B. $P(x) = 4x^3 - 20x^2$
C. $P(x) = 6x^2 - 20x + 5$
D. $P(x) = 8x^4 - 40x^3 - 20x^2 - 100x$

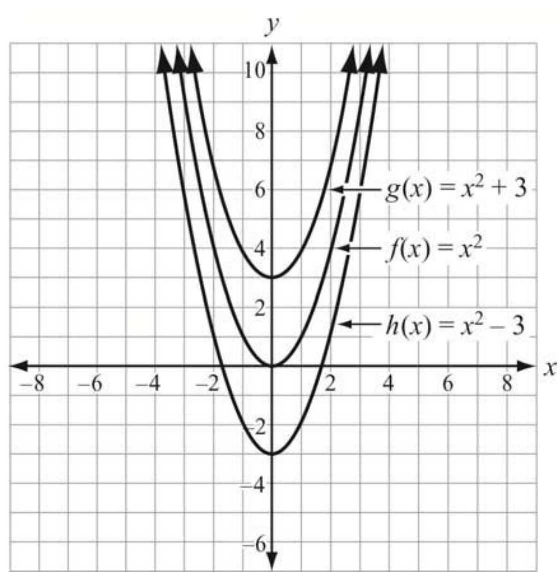
Correct Answer: A

Build New Functions from Existing Functions

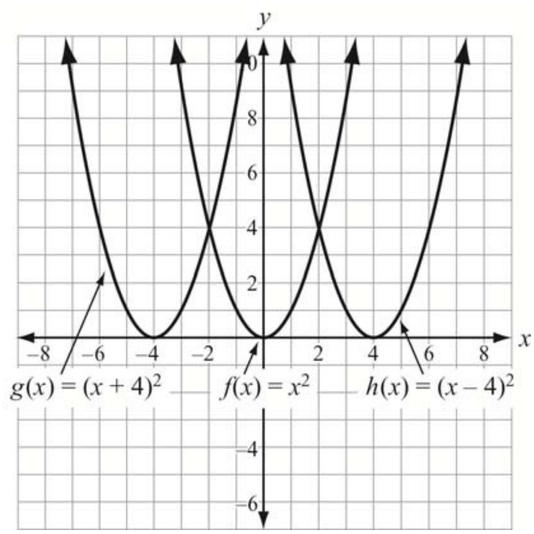
MGSE9-12.F.BF.3 Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k\,f(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them.

KEY IDEAS

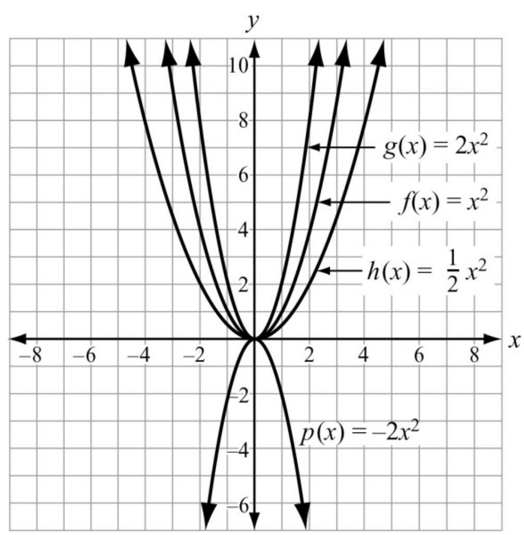
1. A ***parent function*** is the basic function from which all the other functions in a function family are modeled. For the quadratic function family, the parent function is $f(x) = x^2$.
2. For a parent function $f(x)$ and a real number k,
   - the function $f(x) + k$ will move the graph of $f(x)$ up by k units.
   - the function $f(x) – k$ will move the graph of $f(x)$ down by k units.



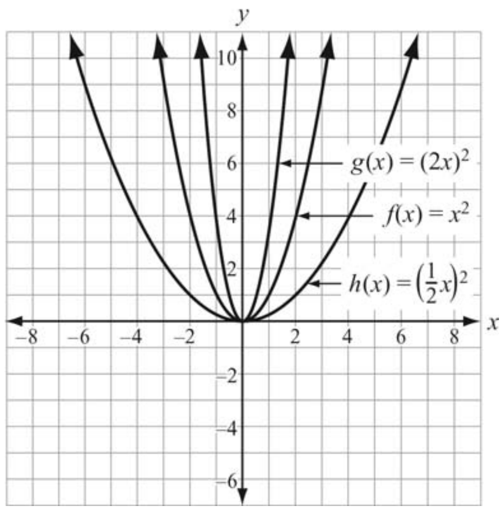
3. For a parent function $f(x)$ and a real number k,
   - the function $f(x + k)$ will move the graph of $f(x)$ left by k units.
   - the function $f(x - k)$ will move the graph of $f(x)$ right by k units.



4. For a parent function $f(x)$ and a real number k,
   - the function $kf(x)$ will vertically stretch the graph of $f(x)$ by a factor of k units for $|k| > 1$.
   - the function $kf(x)$ will vertically shrink the graph of $f(x)$ by a factor of k units for $|k| < 1$.
   - the function $kf(x)$ will reflect the graph of $f(x)$ over the x-axis for negative values of k.

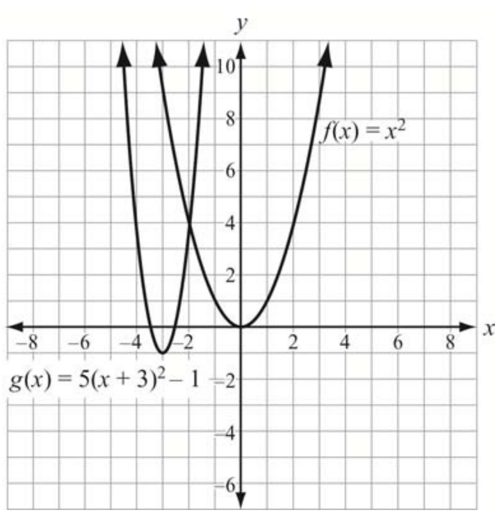


5. For a parent function $f(x)$ and a real number k,
   - the function $f(kx)$ will horizontally shrink the graph of $f(x)$ by a factor of $\frac{1}{k}$ units for $|k| > 1$.
   - the function $f(kx)$ will horizontally stretch the graph of $f(x)$ by a factor of $\frac{1}{k}$ units for $|k| < 1$.
   - the function $f(kx)$ will reflect the graph of $f(x)$ over the y-axis for negative values of k.

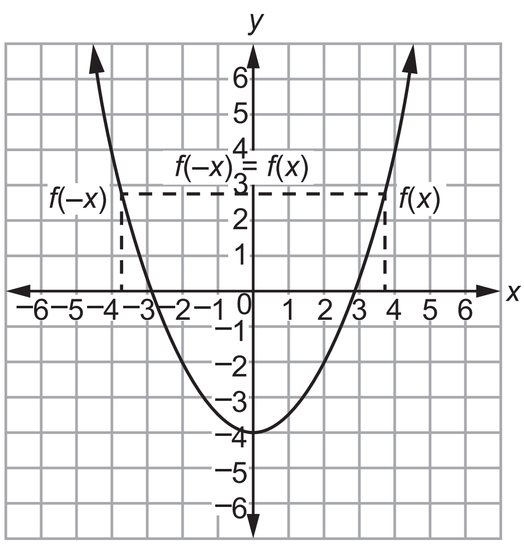


6. You can apply more than one of these changes at a time to a parent function.

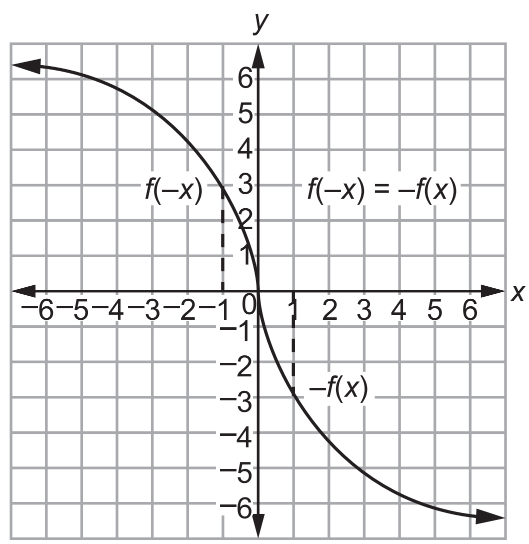
   Example:
   $f(x) = 5(x + 3)^2 - 1$ will translate $f(x) = x^2$ left 3 units and down 1 unit and stretch the function vertically by a factor of 5.



7. Functions can be classified as even or odd.
   - If a graph is symmetric to the y-axis, then it is an ***even function***.
     That is, if $f(-x) = f(x)$, then the function is even.
   - If a graph is symmetric to the origin, then it is an ***odd function***.
     That is, if $f(-x) = -f(x)$, then the function is odd.



Symmetric about the y-axis
Even Function



Symmetric about the origin
Odd Function

Important Tip

✍ Remember that when you change $f(x)$ to $f(x + k)$, move the graph to the **left** when k is positive and to the **right** when k is negative. This may seem different from what you would expect, so be sure to understand why this occurs in order to apply the shift or translation correctly.

REVIEW EXAMPLES

1. Compare the graphs of the following functions to $f(x)$.

   a. $\frac{1}{2}f(x)$

   b. $f(x) - 5$

   c. $f(x - 2) + 1$

Solution:

a. The graph of $\frac{1}{2}f(x)$ is a vertical shrink of $f(x)$ by a factor of $\frac{1}{2}$.

b. The graph of $f(x) - 5$ is a shift or vertical translation of the graph of $f(x)$ down 5 units.

c. The graph of $f(x - 2) + 1$ is a shift or vertical translation of the graph of $f(x)$ right 2 units and up 1 unit.

2. Is $f(x) = 2x^3 + 6x$ even, odd, or neither? Explain how you know.

Solution:

Substitute $-x$ for x and evaluate:

$$\begin{aligned} f(-x) &= 2(-x)^3 + 6(-x) \\ &= 2(-x)^3 - 6x \\ &= -(2x^3 + 6x) \end{aligned}$$

$f(-x)$ is the opposite of $f(x)$, so the function is odd.

Substitute -3 for x and evaluate:

$$\begin{aligned} f(-3) &= 2(-3)^3 + 6(-3) \\ &= 2(-3)^3 - 18 \\ &= -(2(27) + 18) \\ &= -(72) \end{aligned} \qquad \begin{aligned} f(3) &= 2(3)^3 + 6(3) \\ &= 2(27) + 18 \\ &= 72 \end{aligned}$$

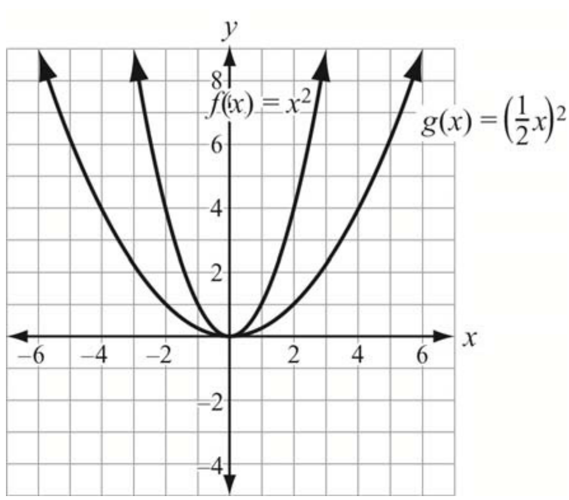
$f(-3)$ is the opposite of $f(3)$, so the function is odd.

3. How does the graph of $f(x)$ compare to the graph of $f\left(\frac{1}{2}x\right)$?

Solution:

The graph of $f\left(\frac{1}{2}x\right)$ is a horizontal stretch of $f(x)$ by a factor of 2. The graphs of $f(x)$ and $g(x) = f\left(\frac{1}{2}x\right)$ are shown.

For example, at $y = 4$, the width of $f(x)$ is 4 and the width of $g(x)$ is 8. So, the graph of $g(x)$ is wider than $f(x)$ by a factor of 2.



SAMPLE ITEMS

1. Which statement BEST describes the graph of $f(x + 6)$?

A. The graph of $f(x)$ is shifted up 6 units.
B. The graph of $f(x)$ is shifted left 6 units.
C. The graph of $f(x)$ is shifted right 6 units.
D. The graph of $f(x)$ is shifted down 6 units.

Correct Answer: B

2. Which of these is an even function?

A. $f(x) = 5x^2 - x$
B. $f(x) = 3x^3 + x$
C. $f(x) = 6x^2 - 8$
D. $f(x) = 4x^3 + 2x^2$

Correct Answer: C

3. Which statement BEST describes how the graph of $g(x) = -3x^2$ compares to the graph of $f(x) = x^2$?

A. The graph of $g(x)$ is a vertical stretch of $f(x)$ by a factor of 3.
B. The graph of $g(x)$ is a reflection of $f(x)$ across the x-axis.
C. The graph of $g(x)$ is a vertical shrink of $f(x)$ by a factor of $\frac{1}{3}$ and a reflection across the x-axis.
D. The graph of $g(x)$ is a vertical stretch of $f(x)$ by a factor of 3 and a reflection across the x-axis.

Correct Answer: D

Interpret Functions That Arise in Applications in Terms of the Context

MGSE9-12.F.IF.4 Using tables, graphs, and verbal descriptions, interpret the key characteristics of a function which models the relationship between two quantities. Sketch a graph showing key features including: intercepts; interval where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior.

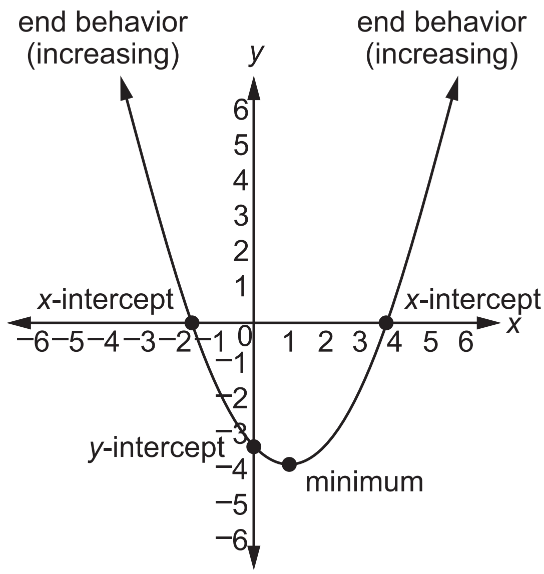
MGSE9-12.F.IF.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*

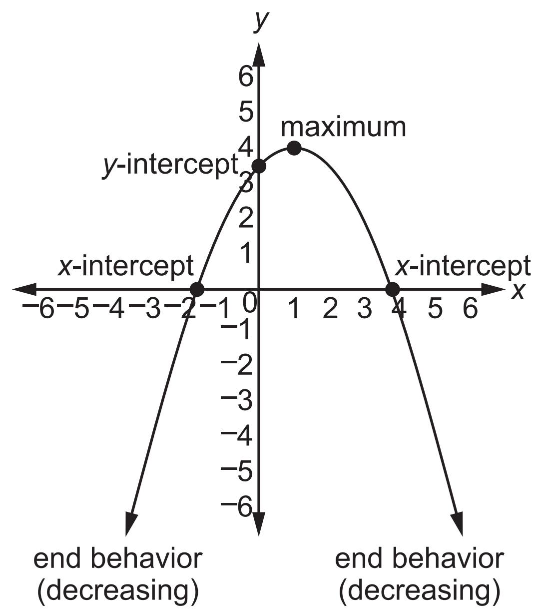
MGSE9-12.F.IF.6 Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.

KEY IDEAS

1. An ***x-intercept***, ***root***, or ***zero*** of a function is the *x*-coordinate of a point where the function crosses the *x*-axis. A function may have multiple *x*-intercepts. To find the *x*-intercepts of a quadratic function, set the function equal to 0 and solve for *x*. This can be done by factoring, completing the square, or using the quadratic formula.
2. The ***y-intercept*** of a function is the *y*-coordinate of the point where the function crosses the *y*-axis. A function may have zero *y*-intercepts or one *y*-intercept. To find the *y*-intercept of a quadratic function, find the value of the function when *x* equals 0.
3. A function is ***increasing*** over an interval when the values of *y* increase as the values of *x* increase over that interval. The interval is represented in terms of *x*.
4. A function is ***decreasing*** over an interval when the values of *y* decrease as the values of *x* increase over that interval. The interval is represented in terms of *x*.
5. Every quadratic function has a ***minimum*** or ***maximum***, which is located at the vertex. When the function is written in standard form, the *x*-coordinate of the vertex is $\frac{-b}{2a}$. To find the *y*-coordinate of the vertex, substitute the value of $\frac{-b}{2a}$ into the function and evaluate. When the value of *a* is positive, the graph opens up, and the vertex is the minimum point. When the value of *a* is negative, the graph opens down, and the vertex is the maximum point.
6. The ***end behavior*** of a function describes how the values of the function change as the *x*-values approach negative infinity and positive infinity.
7. The ***domain*** of a function is the set of values for which it is possible to evaluate the function. The domain of a quadratic function is typically all real numbers, although in real-world applications it may only make sense to look at the domain values on a particular interval. For example, time must be a non-negative number.

8. The ***average rate of change*** of a function over a specified interval is the change in the *y*-value divided by the change in the *x*-value for two distinct points on a graph. To calculate the average rate of change of a function over the interval from *a* to *b*, evaluate the expression $m = \frac{y_2 - y_1}{x_2 - x_1}$.





REVIEW EXAMPLES

1. A ball is thrown into the air from a height of 4 feet at time $t = 0$. The function that models this situation is $h(t) = -16t^2 + 63t + 4$, where t is measured in seconds and h is the height in feet.

   a. What is the height of the ball after 2 seconds?

   b. When will the ball reach a height of 50 feet?

   c. What is the maximum height of the ball?

   d. When will the ball hit the ground?

   e. What domain makes sense for the function?

Solution:

a. To find the height of the ball after 2 seconds, substitute 2 for t in the function.

$h(2) = -16(2)^2 + 63(2) + 4 = -16(4) + 126 + 4 = -64 + 126 + 4 = 66$

The height of the ball after 2 seconds is 66 feet.

b. To find when the ball will reach a height of 50 feet, find the value of t that makes $h(t) = 50$.

$50 = -16t^2 + 63t + 4$

$0 = -16t^2 + 63t - 46$

Use the quadratic formula with $a = -16$, $b = 63$, and $c = -46$.

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$t = \frac{-63 \pm \sqrt{(63)^2 - 4(-16)(-46)}}{2(-16)}$$

$$t = \frac{-63 \pm \sqrt{3969 - 2944}}{-32}$$

$$t = \frac{-63 \pm \sqrt{1025}}{-32}$$

$t \approx 0.97$ or $t \approx 2.97$. So, the ball is at a height of 50 feet after approximately 0.97 second and 2.97 seconds.

c. To find the maximum height, find the vertex of $h(t)$.

The x-coordinate of the vertex is equal to $\frac{-b}{2a}$: $\frac{-63}{2(-16)} \approx 1.97$. To find the y-coordinate, find $h(1.97)$:

$$\begin{aligned} h(1.97) &= -16(1.97)^2 + 63(1.97) + 4 \\ &\approx 66 \end{aligned}$$

The maximum height of the ball is about 66 feet.

d. To find when the ball will hit the ground, find the value of t that makes $h(t) = 0$ (because 0 represents 0 feet from the ground).

$0 = -16t^2 + 63t + 4$

Using the quadratic formula (or by factoring), $t = -0.0625$ or $t = 4$.

Time cannot be negative, so $t = -0.0625$ is not a solution. The ball will hit the ground after 4 seconds.

e. Time must always be non-negative and can be expressed by any fraction or decimal. The ball is thrown at 0 seconds and reaches the ground after 4 seconds. So, the domain $0 \leq t \leq 4$ makes sense for function $h(t)$.

2. This table shows a company's profit, p, in thousands of dollars, over time, t, in months.

Time, *t* (months)	Profit, *p* (thousands of dollars)
3	18
7	66
10	123
15	258
24	627

a. Describe the average rate of change in terms of the given context.
b. What is the average rate of change of the profit between 3 and 7 months?
c. What is the average rate of change of the profit between 3 and 24 months?

Solution:

a. The average rate of change represents the rate at which the company earns a profit.

b. Use the expression for average rate of change. Let $x_1 = 3$, $x_2 = 7$, $y_1 = 18$, and $y_2 = 66$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{66 - 18}{7 - 3} = \frac{48}{4} = 12$$

The average rate of change between 3 and 7 months is 12 thousand dollars ($12,000) per month.

c. Use the expression for average rate of change. Let $x_1 = 3$, $x_2 = 24$, $y_1 = 18$, and $y_2 = 627$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{627 - 18}{24 - 3} = \frac{609}{21} = 29$$

The average rate of change between 3 and 24 months is 29 thousand dollars ($29,000) per month.

SAMPLE ITEMS

1. A flying disk is thrown into the air from a height of 25 feet at time $t = 0$. The function that models this situation is $h(t) = -16t^2 + 75t + 25$, where t is measured in seconds and h is the height in feet. What values of t best describe the times when the disk is flying in the air?

A. $0 < t < 5$
B. $0 < t < 25$
C. all real numbers
D. all positive integers

Correct Answer: A

2. Use this table to answer the question.

x	$f(x)$
–2	15
–1	9
0	5
1	3
2	3

What is the average rate of change of $f(x)$ over the interval $-2 \leq f(x) \leq 0$?

A. –10
B. –5
C. 5
D. 10

Correct Answer: B

3. What is the end behavior of the graph of $f(x) = -0.25x^2 - 2x + 1$?

A. As x increases, $f(x)$ increases.
As x decreases, $f(x)$ decreases.

B. As x increases, $f(x)$ decreases.
As x decreases, $f(x)$ decreases.

C. As x increases, $f(x)$ increases.
As x decreases, $f(x)$ increases.

D. As x increases, $f(x)$ decreases.
As x decreases, $f(x)$ increases.

Correct Answer: B

Analyze Functions Using Different Representations

MGSE9-12.F.IF.7 Graph functions expressed algebraically and show key features of the graph both by hand and by using technology.

MGSE9-12.F.IF.7a Graph linear and quadratic functions and show intercepts, maxima, and minima (as determined by the function or by context).

MGSE9-12.F.IF.8 Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.

MGSE9-12.F.IF.8a Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context. *For example, compare and contrast quadratic functions in standard, vertex, and intercept forms.*

MGSE9-12.F.IF.9 Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). For example, given a graph of one function and an algebraic expression for another, say which has the larger maximum.

KEY IDEAS

1. Functions can be represented algebraically, graphically, numerically (in tables), or verbally (by description).

   Examples:

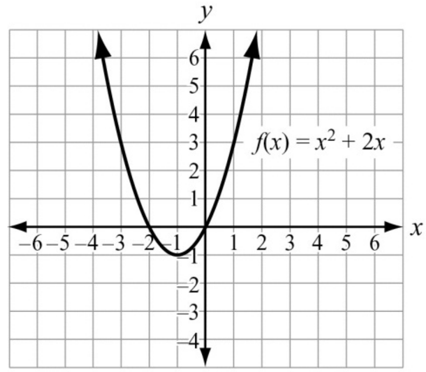
   Algebraically: $f(x) = x^2 + 2x$

   Verbally (by description): a function that represents the sum of the square of a number and twice the number

   Numerically (in a table):

x	$f(x)$
–1	–1
0	0
1	3
2	8

Graphically:



2. You can compare key features of two functions represented in different ways. For example, if you are given an equation of a quadratic function and a graph of another quadratic function, you can calculate the vertex of the first function and compare it to the vertex of the graphed function.

REVIEW EXAMPLES

1. Graph the function $f(x) = x^2 - 5x - 24$.

Solution:

Use the algebraic representation of the function to find the key features of the graph of the function.

Find the zeros of the function.

$$0 = x^2 - 5x - 24 \quad \text{Set the function equal to 0.}$$
$$0 = (x - 8)(x + 3) \quad \text{Factor.}$$

Set each factor equal to 0 and solve for x.

$$x - 8 = 0 \qquad x + 3 = 0$$
$$x = 8 \qquad x = -3$$

The zeros are at $x = -3$ and $x = 8$.

Find the vertex of the function.

$$x = \frac{-b}{2a} = \frac{-(-5)}{2(1)} = \frac{5}{2} = 2.5$$

Substitute 2.5 for x in the original function to find $f(2.5)$:

$$f(x) = x^2 - 5x - 24$$
$$f(2.5) = (2.5)^2 - 5(2.5) - 24 = 6.25 - 12.5 - 24 = -30.25$$

The vertex is (2.5, –30.25).

Find the y-intercept by finding $f(0)$.

$$f(x) = x^2 - 5x - 24$$
$$f(0) = (0)^2 - 5(0) - 24 = -24$$

The *y*-intercept is (0, –24). Use symmetry to find another point. The line of symmetry is $x = 2.5$.

$$\frac{0 + x}{2} = 2.5$$

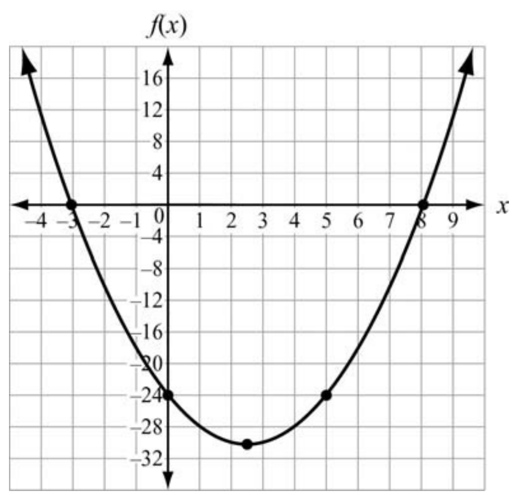
$$0 + x = 5$$

$$x = 5$$

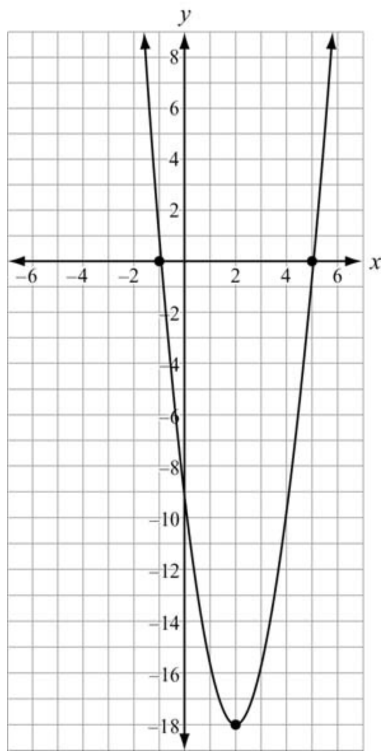
So, point (5, –24) is also on the graph.

Plot the points (–3, 0), (8, 0), (2.5, –30.25), (0, –24), and (5, –24). Draw a smooth curve through the points.

We can also use the value of *a* in the function to determine if the graph opens up or down. In $f(x) = x^2 - 5x - 24$, $a = 1$. Since $a > 0$, the graph opens up.



2. This graph shows a function $f(x)$.



Compare the graph of $f(x)$ to the graph of the function given by the equation $g(x) = 4x^2 + 6x - 18$. Which function has the lesser minimum value? How do you know?

Solution:

The minimum value of a quadratic function is the y-value of the vertex.

The vertex of the graph of $f(x)$ appears to be (2, –18). So, the minimum value is –18.

Find the vertex of the function $g(x) = 4x^2 + 6x - 18$.

To find the vertex of $g(x)$, use $\left(\frac{-b}{2a}, g\left(\frac{-b}{2a}\right)\right)$ with $a = 4$ and $b = 6$.

$$x = \frac{-b}{2a} = \frac{-(6)}{2(4)} = \frac{-6}{8} = -0.75$$

Substitute –0.75 for x in the original function $g(x)$ to find $g(-0.75)$:

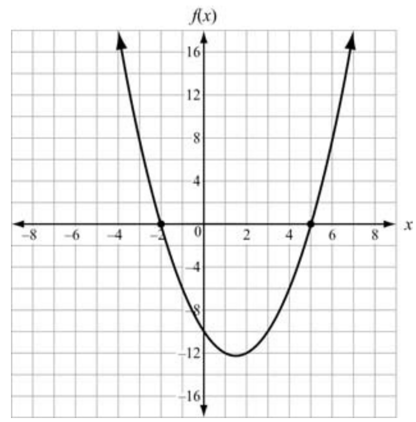
$$\begin{aligned} g(x) &= 4x^2 + 6x - 18 \\ g(-0.75) &= 4(-0.75)^2 + 6(-0.75) - 18 \\ &= 2.25 - 4.5 - 18 \\ &= -20.25 \end{aligned}$$

The minimum value of $g(x)$ is –20.25.

$-20.25 < -18$, so the function $g(x)$ has the lesser minimum value.

SAMPLE ITEMS

1. Use this graph to answer the question.



Which function is shown in the graph?

A. $f(x) = x^2 - 3x - 10$
B. $f(x) = x^2 + 3x - 10$
C. $f(x) = x^2 + x - 12$
D. $f(x) = x^2 - 5x - 8$

Correct Answer: A

2. The function $f(t) = -16t^2 + 64t + 5$ models the height of a ball that was hit into the air, where t is measured in seconds and h is the height in feet.

This table represents the height, $g(t)$, of a second ball that was thrown into the air.

Time, t (in seconds)	Height, $g(t)$ (in feet)
0	4
1	36
2	36
3	4

Which statement BEST compares the length of time each ball is in the air?

A. The ball represented by $f(t)$ is in the air for about 5 seconds, and the ball represented by $g(t)$ is in the air for about 3 seconds.

B. The ball represented by $f(t)$ is in the air for about 3 seconds, and the ball represented by $g(t)$ is in the air for about 5 seconds.

C. The ball represented by $f(t)$ is in the air for about 3 seconds, and the ball represented by $g(t)$ is in the air for about 4 seconds.

D. The ball represented by $f(t)$ is in the air for about 4 seconds, and the ball represented by $g(t)$ is in the air for about 3 seconds.

Correct Answer: D